THE KAUFFMAN BRACKET AND THE BOLLOBÁS-RIORDAN POLYNOMIAL OF RIBBON GRAPHS

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ABSTRACT. For a ribbon graph G we consider an alternating link L_G in the 3-manifold $G \times I$ represented as the product of the oriented surface G and the unit interval I. We show that the Kauffman bracket $[L_G]$ is an evaluation of the recently introduced Bollobás-Riordan polynomial R_G. This results generalizes the celebrated relation between Kauffman bracket and Tutte polynomial of planar graphs.

INTRODUCTION

The study of polynomial invariants was revolutionized after Jones' celebrated discovery of Jones polynomials of knots and links in $\mathbb{R}^3$. In [K1] L. Kauffman showed how to avoid Jones' somewhat involved algebraic tools, and presented a simple combinatorial definition of a more general polynomial, now called the Kauffman bracket, associated to (signed) planar graphs, of which Jones polynomial is an evaluation. In the same paper Kauffman also showed that this polynomial is in fact equivalent to the (signed) Tutte polynomial well studied in combinatorics literature.

The study of the Kauffman bracket, the Jones polynomial, and other link polynomials was soon extended to links in $G \times I$, the products of a two-dimensional surface G and the interval $I = [0, 1]$; we refer to [CR, HP, IK, Tu] and a survey [Pr] for various approaches to the subject. In recent years the Kauffman bracket of links in $G \times I$ received additional attention motivated by new results on crossing numbers [AFLT] and the study of virtual knots [DK, M]. All these papers utilize and develop the first half of "Kauffman's program", that is they extend Kauffman's combinatorial approach to obtain the generalized Jones and other polynomial invariants of knots and links in $G \times I$.

In this paper we are concerned with the second half of the "Kauffman's program", on establishing the connection between the Kauffman bracket and the Tutte polynomial. We show that the Kauffman bracket of links in $G \times I$ is an evaluation of the recently introduced Bollobás-Riordan polynomials of ribbon graphs. The latter generalize the classical (dichromatic) Tutte polynomial from general graphs to graphs on surfaces, and our results is an extension of Kauffman's relation. Interestingly, the Bollobás-Riordan polynomials were also introduced with (very different) knot theoretic applications in mind [BR2, BR3].

The paper is structured as follows. In the first two sections we recall definitions of Kauffman's bracket and the Bollobás-Riordan polynomial of ribbon graphs. In section 3 we construct 'medial ribbon graphs' and state the Main Theorem. As often appears in these cases, the proofs of results on (generalizations of) the Tutte polynomial are quite straightforward, so the proof of the Main Theorem is postponed till section 5. In section 4

we extend our results to signed ribbon graphs and derive the Jones polynomial of links in $G \times I$ as an appropriate evaluation. We conclude with final remarks and overview of the literature.

1. The Kauffman bracket in $G \times I$.

Let G be an oriented surface (possibly with the boundary), let $I = [0, 1]$ be the unit interval, and let L be an unoriented link in the 3-manifold $G \times I$. To represent L by its diagram on the surface G we will always assume that L is in general position with respect to the projection $\pi \colon G \times I \to G$, i.e. the image $\pi(L)$ is an immersed curve in G with finitely many double points as its only singularities. This can be achieved by a small deformation of L which preserves the topological type of the link L. Then a *diagram* of L, denotes $\widetilde{L}$ is the curve $\pi(L) \subset G$ with an extra information of *overcrossing* and *undercrossing* at every double point of $\pi(L)$. Naturally, the link L can have many different diagrams $\widetilde{L}$.

Consider two ways of resolving a crossing in L. The *A-splitting*, $\dashv\mathrel{\rightsquigarrow}$ ⌐, is obtained by uniting two regions swept out by the overcrossing arc under the counterclockwise rotation until the undercrossing arc. We are assuming here that the orientation of G is given by the counterclockwise rotation, so the overcrossing arc was rotated according to the orientation. Similarly, the *B-splitting*, $\dashv\mathrel{\rightsquigarrow}$ ⌐, is obtained by uniting two other regions. A *state* S of the link diagram $\widetilde{L}$ is a way of splitting at each crossing of the diagram, and denote by $\mathcal{S}(\widetilde{L})$ the set of such states. Clearly, a diagram $\widetilde{L}$ with n crossings has $|\mathcal{S}(\widetilde{L})| = 2^n$ different states.
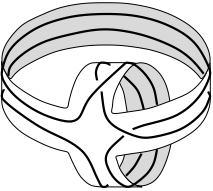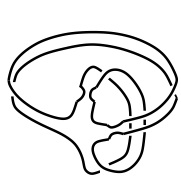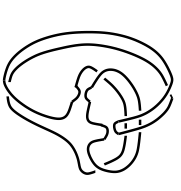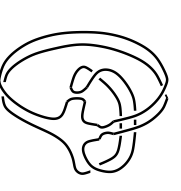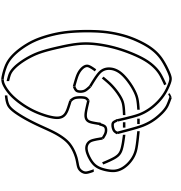
Denote by $\alpha(S)$ and $\beta(S)$ the number of A-splittings and B-splittings in a state S, respectively. Also, denote by $\delta(S)$ the number of connected components of the curve obtained from the link diagram $\widetilde{L}$ by all splittings according to state $S \in \mathcal{S}(\widetilde{L})$.

Definition 1.1. *The* Kauffman bracket *of a diagram $\widetilde{L}$ of a link $L \subset G \times I$ is a polynomial in three variables A, B, d defined by the formula:*

$$(1) \qquad [\widetilde{L}](A, B, d) \ := \ \sum_{S \in \mathcal{S}(\tilde{L})} A^{\alpha(S)} \, B^{\beta(S)} \, d^{\delta(S)-1} \,.$$

This definition of the (generalized) Kauffman's bracket follows [Ka]. Note that $[\widetilde{L}]$ is *not* a topological invariant of the link L and in fact depends on the link diagram $\widetilde{L}$.

Example 1.2. Consider the surface G and the link $L \subset G \times I$ as shown on the left in the table below. The corresponding diagram $\widetilde{L}$ has two crossings, so there are four states for it, $|\mathcal{S}(\widetilde{L})| = 4$. The curves obtained by the splittings and the corresponding parameters $\alpha(S)$, $\beta(S)$, and $\delta(S)$ are shown in the remaining columns of the table.

| $(\alpha(S), \beta(S), \delta(S))$ | $(2,0,1)$ | $(1,1,2)$ | $(1,1,2)$ | $(0,2,1)$ |

In this case the Kauffman bracket of $\widetilde{L}$ is given by $[\widetilde{L}] = A^2 + 2ABd + B^2$.

2. The Bollobás-Riordan polynomial.

Let $\Gamma = (V, E)$ be a undirected graph with the set of vertices V and the set of edges E (loops and multiple edges are allowed). Suppose in each vertex $v \in V$ there is a fixed cyclic order on edges adjacent to v (loops are counted twice). We call this combinatorial structure the *ribbon graph*, and denote it by G. One can represent G by making vertices into 'discs' and connecting them by 'ribbons' as prescribed by the cyclic orders (see Example 2.2 below). This defines a 2-dimensional surface with the boundary, which by a slight abuse of notation we also denote by G.

Formally, G is the surface with the boundary represented as the union of two sets of closed topological disks, corresponding to vertices $v \in V$ and edges $e \in E$, which satisfies the following conditions:

- these discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge,
- every edge contains exactly two such line segments.

It will be clear from the context whether by G we mean the ribbon graph or its underlying surface. In this paper we restrict ourselves to oriented surfaces G. We refer to [GT] for other definitions and references. We use the following standard notation.

For a ribbon graph G let $v(G) = |V|$ denotes the number of vertices, $e(G) = |E|$ denotes the number of edges, and let $k(G)$ be the number of connected components of G. Also, let $r(G) = v(G) - k(G)$ be the *rank* of G, and let $n(G) = e(G) - r(G)$ be the *nullity* of G. Finally, let $\mathrm{bc}(G)$ be the number of connected components of the boundary of the surface G.

A spanning subgraph of a ribbon graph G is defined as a subgraph which contains all the vertices, and a subset of edges. Let $\mathcal{F}(G)$ denote the set of spanning subgraphs of G. Clearly, the number of spanning subgraphs $F \subseteq G$ is equal to $|\mathcal{F}(G)| = 2^{e(G)}$.

Definition 2.1. *The* Bollobás-Riordan *polynomial $R_G(x, y, z)$ of a ribbon graph G is defined by the formula*

$$(2) \qquad R_G(x, y, z) \; := \; \sum_{F \in \mathcal{F}(G)} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-\mathrm{bc}(F)+n(F)} \, ,$$

where the sum runs over all spanning subgraphs F of G.

This version of the polynomial is obtained from the original one [BR2, BR3] by a simple substitution. Note that for all planar ribbon graphs $F \subseteq G$ (i.e. when the surface F has genus zero) the Euler formula gives $k(F) - \mathrm{bc}(F) + n(F) = 0$. So, for a planar ribbon graph G the Bollobás-Riordan polynomial R_G does not contain powers of z. In fact, in this case it is essentially equal to the classical Tutte polynomial $T_\Gamma(x, y)$ of the (abstract) graph Γ:
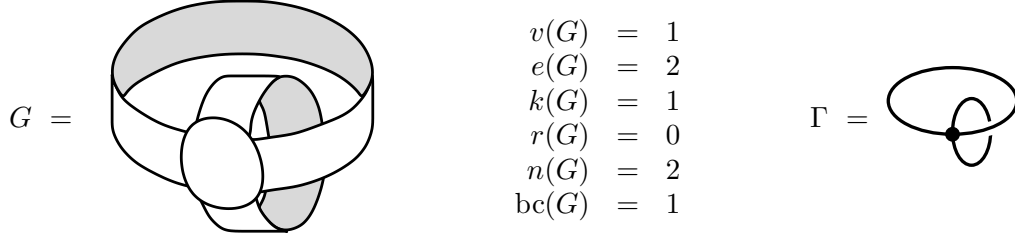
$$R_G(x - 1, y - 1, z) = T_\Gamma(x, y) \, .$$

Similarly, a specialization $z = 1$ of the Bollobás-Riordan polynomial of arbitrary ribbon graph G, gives the Tutte polynomial once again:

$$R_G(x - 1, y - 1, 1) = T_\Gamma(x, y) \, .$$

We refer to [BR2, BR3] for proofs of these formulas and to [B, W] for general background on the Tutte polynomial.

Example 2.2. Consider the following ribbon graph G with the surface as in the Example 1.2, which corresponds to the abstract graph Γ as below.

$$
\begin{array}{rcl}
v(G) & = & 1 \\
e(G) & = & 2 \\
k(G) & = & 1 \\
r(G) & = & 0 \\
n(G) & = & 2 \\
\mathrm{bc}(G) & = & 1
\end{array}
$$

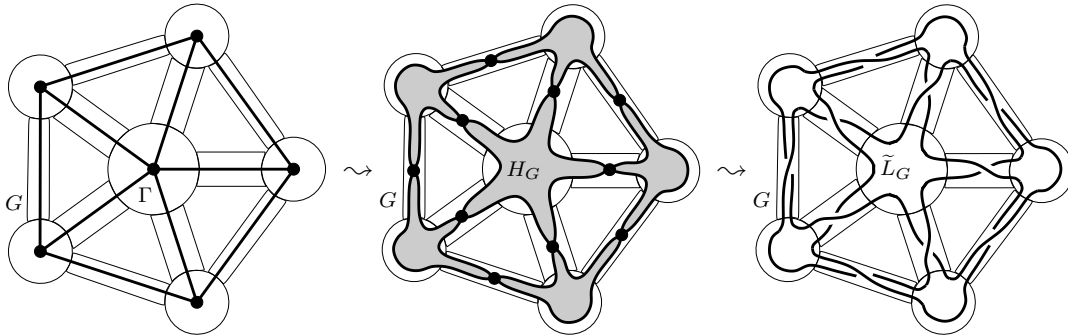By definition, one can compute the corresponding Bollobás-Riordan polynomial:

$$R_G(x, y, z) \; = \; 1 + 2y + y^2 z^2 \,.$$

3. Medial alternating links and the Main Theorem.

The main result of this paper is a connection between the Bollobás-Riordan polynomial $R_G(x, y, z)$ and the Kauffman bracket $[\widetilde{L}_G](A, B, d)$ of a medial alternating link $L_G \in G \times I$ defined below. This connection naturally generalizes Kauffman's result for planar graphs.

Let G be a ribbon graph and Γ be the corresponding abstract graph embedded into the surface G. We construct the (ribbon) *medial graph* H_G by embedding into the surface G as follows. The vertices of H_G lie in the middle of the edges of Γ; every vertex has valence 4. The edges of H_G go along the edge-ribbon of G until the intersection with the vertex-disks where they turn to the next edge-ribbon (see the figure below).

Now, consider the *chess-board coloring* of the regions of H_G: color a region *black* if it contains a vertex of Γ; otherwise color it *white*. We define the link $L_G \subset G \times I$ as follows. Make a link diagram $\widetilde{L}_G$ out of H_G by making a crossing at each vertex of H_G in such a way that the overcrossing branch sweeps out the black regions in our coloring when rotated according to the orientation of the surface G until the undercrossing branch. See an example below, where the orientation of G is given by the counterclockwise rotation:

This construction gives the *natural diagram* $\widetilde{L}_G$ of the *medial alternating link* L_G for the ribbon graph G. Note that the image of L_G under the projection $G \times I \to G$ is the medial graph H_G. Also, the medial alternating link L_G for the ribbon graph in Example 2.2 is precisely the link L in Example 1.2.

Main Theorem 3.1. *For every ribbon graph G and the corresponding medial alternating link $L_G \subset G \times I$ with the natural diagram $\widetilde{L}_G$ we have:*

$$[\widetilde{L}_G](A, B, d) \ = \ A^{r(G)} B^{n(G)} d^{k(G)-1} R_G \left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d} \right).$$

We should warn the reader that although both Kauffman bracket and Bollobás-Riordan polynomial are polynomials in three variables, the former has only two free variables since $[\widetilde{L}]$ is always homogeneous in A and B. This follows from $\alpha(S) + \beta(S) = e(S)$ for all $S \in \mathcal{S}(\widetilde{L})$. Another way to see this is to note that the values of the Bollobás-Riordan polynomial $R_G(x, y, z)$ evaluated in the theorem satisfy $xyz^2 = 1$ equation. So the situation here is different from the planar case where the Kauffman bracket $[\widetilde{L}_G](A, B, d)$ and the Tutte polynonomial $T_\Gamma(x, y)$ detrmine each other.

4. EXTENSIONS AND APPLICATIONS.

The *signed ribbon graph* $\widehat{G}$ is a ribbon graph G given by (V, E) with a *sign function* $\varepsilon : E \to \{\pm 1\}$. For a spanning subgraph $F \subset G$ denote by $e_-(F)$ the number of edges $e \in E$ with $\varepsilon(e) = -1$. Denote by $\overline{F} = G - F$ the complementary spanning subgraph of G with only those edges of G that do not belong to F. Finally, let $s(F) = \frac{1}{2}\big(e_-(F) - e_-(\overline{F})\big)$.

We define the *signed Bollobás-Riordan polynomial* $R_{\widehat{G}}(x, y, z)$ as follows:

$$(3) \qquad R_{\widehat{G}}(x, y, z) \ := \ \sum_{F \in \mathcal{F}(G)} x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}.$$

Similarly, define the *signed medial link* $L_{\widehat{G}}$ by a diagram obtained from $\widetilde{L}_G$ by switching the overcrossings to undercrossings for negative edges of $\widehat{G}$.

Theorem 4.1. *For every signed ribbon graph $\widehat{G}$ and the corresponding signed medial link $L_{\widehat{G}} \subset G \times I$ with the diagram $\widetilde{L} = \widetilde{L}_{\widehat{G}}$ we have:*

$$[\widetilde{L}](A, B, d) \ = \ A^{r(G)} B^{n(G)} d^{k(G)-1} R_{\widehat{G}} \left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d} \right).$$

The proof is follows verbatim the proof of the Main Theorem (see the next section). We leave the details to the reader.

Now recall that the Jones polynomial is defined for oriented links. It can be obtained evaluating the Kauffman bracket [K1]:

$$J_L(t) \ := \ (-1)^{w(\widetilde{L})} t^{3w(\widetilde{L})/4} [\widetilde{L}](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}),$$

where $w(\widetilde{L})$ is the *writhe* of a diagram $\widetilde{L}$ determined by the orientation of $\widetilde{L}$ as the sum of the following signs of the crossings of $\widetilde{L}$:

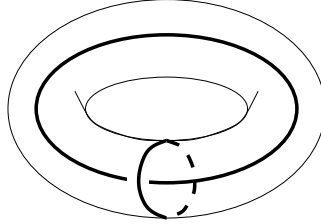The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let $\widehat{G}$ be a signed ribbon graph and let $L = L_{\widehat{G}}$ be the corresponding signed medial link endowed with an orientation. Then*

$$J_L(t) = (-1)^{w(\widetilde{L})} t^{\frac{3w(\widetilde{L})-r(\widehat{G})+n(\widehat{G})}{4}} \left(-t^{1/2}-t^{-1/2}\right)^{k(\widehat{G})-1} R_{\widehat{G}}\left(-t-1,-t^{-1}-1,\frac{1}{-t^{1/2}-t^{-1/2}}\right).$$

In particular, if $\widehat{G}$ is a planar ribbon graph with only positive edges and with underlying graph Γ, we have the following well-known relation:

$$J_L(t) = (-1)^{w(\widetilde{L})} t^{\frac{3w(\widetilde{L})-r(\widehat{G})+n(\widehat{G})}{4}} \left(-t^{1/2}-t^{-1/2}\right)^{k(\widehat{G})-1} T_\Gamma(-t,-t^{-1}).$$

Remark 4.3. In contrast with the (usual) links in $\mathbb{R}^3$, not every link in the product of a surface G and the interval I can be represented as a signed medial link of a subsurface of G. The simplest example is the link in the product of the torus and I, as shown in figure below. Thus Theorem 4.1 and Corollary 4.2 cannot be used to compute the Kauffman bracket and the Jones polynomial of an arbitrary link $L \in G \times I$.

5. Proof of the Main Theorem.

The notation used in the definitions of the Kauffman bracket (Definition 1.1) and the Bollobás-Riordan polynomial (Definition 2.1) suggests a hint on how to prove the Main Theorem. There is a natural one-to-one correspondence $\varphi : \mathcal{S}(\widetilde{L}_G) \to \mathcal{F}(G)$ between the states $S \in \mathcal{S}(\widetilde{L}_G)$ and spanning subgraphs $F \subseteq G$. Indeed, note that the crossings of the diagram $\widetilde{L}_G$ correspond to the edges of G. Now, let an A-splitting of a crossing in S mean that we keep the corresponding edge in the spanning subgraph $F = \varphi(S)$. Similarly let a B-splitting in S mean that we remove the edge from the subgraph $F = \varphi(S)$.

By definition, we have $\delta(S) = \mathrm{bc}(F)$, for all $F = \varphi(S)$. Furthermore, we easily obtain the following relation between the parameters:

$$e(F) = \alpha(S), \qquad e(G) - e(F) = \beta(S) ,$$

for all $S \in \mathcal{S}(\widetilde{L}_G)$, and $F = \varphi(S)$. Now, for a spanning subgraph $F \in \mathcal{F}(G)$, consider a term $x^{r(G)-r(F)} y^{n(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$ of $R_G(x,y,z)$. After the substitution

$$x = \frac{Bd}{A}, \qquad y = \frac{Ad}{B}, \qquad z = \frac{1}{d}$$

and multiplication of this term by $A^{r(G)} B^{n(G)} d^{k(G)-1}$ as in the Main Theorem, we get

$$A^{r(G)} B^{n(G)} d^{k(G)-1} (A^{-1}Bd)^{r(G)-r(F)} (AB^{-1}d)^{n(F)} d^{-k(F)+\mathrm{bc}(F)-n(F)}$$

$$= A^{r(G)-r(G)+r(F)+n(F)} B^{n(G)+r(G)-r(F)-n(F)} d^{k(G)-1+r(G)-r(F)+n(F)-k(F)+\mathrm{bc}(F)-n(F)}$$

$$= A^{r(F)+n(F)} B^{n(G)+r(G)-r(F)-n(F)} d^{k(G)-1+r(G)-r(F)-k(F)+\mathrm{bc}(F)} .$$

It is easy to see that $r(F) + n(F) = e(F)$, and $k(F) + r(F) = v(F) = v(G)$. Therefore, $k(G) - k(F) + r(G) - r(F) = 0$, and we can rewrite our term as

$$A^{e(F)} B^{e(G)-e(F)} d^{\mathrm{bc}(F)-1} \ .$$

In terms of the state $S = \varphi^{-1}(F) \in \mathcal{S}(\widetilde{L}_G)$ this summation term is equal to

$$A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1} \ ,$$

which is precisely the term of $[\widetilde{L}_G]$ corresponding to the state $S \in \mathcal{S}(\widetilde{L}_G)$. This completes the proof. $\qquad\square$

6. FINAL REMARKS AND OPEN PROBLEMS.

1. Trivalent ribbon graphs are the main objects in the finite type invariant theory of knots, links and 3-manifolds, while general ribbon graphs appeared in the literature under a variety of different names (see e.g. [DKC, BR2, K1]). Embeddings of ribbon graphs into the 3-space are studied in [RT].

2. The Bollobás-Riordan polynomial can be defined by recurrent contraction-deletion relations or by spanning tree expansion similar to those of the Tutte polynomial, except that deletion by a loop is not allowed. We refer to [BR2, BR3] for the details. We should note that [BR3] gives an extension to unorientable surfaces as well. One can also find the contraction-deletion relations and the spanning tree expansion for the signed Bollobás-Riordan polynomial defined by (3). For a planar signed ribbon graph $\widehat{G}$ the signed Bollobás-Riordan polynomial $R_{\widehat{G}}$ is related to Kauffman's *signed Tutte polynomial* $Q[\widehat{G}]$ from [K2] by the formula

$$R_{\widehat{G}}(x, y, z) = x^{\frac{v(\widehat{G})+1}{2} - k(\widehat{G})} y^{\frac{-v(\widehat{G})+1}{2}} Q[\widehat{G}]\Big((y/x)^{1/2}, 1, (xy)^{1/2} \Big).$$

So our version of $R_{\widehat{G}}$ may be considered as a generalization of the polynomial $Q[\widehat{G}]$ to signed ribbon graphs. If, besides the planarity, all edges of $\widehat{G}$ are positive, then $R_{\widehat{G}}$ is related to the dichromatic polynomial $Z[\Gamma](q, v)$ (see [K2]) of the underlying graph Γ:

$$R_{\widehat{G}}(x, y, z) = x^{-k(\widehat{G})} y^{-v(\widehat{G})} Z[\Gamma](xy, y) \, .$$

3. It would be interesting to generalize the Bollobás-Riordan polynomial for colored ribbon graphs [BR1, Tr] and prove the corresponding relation with the Kauffman bracket. Let us also mention that in [J] (see also [Tr]) Jaeger found a different relation between links and graphs and proved that the whole Tutte polynomial, not just its specialization, can be obtained from the HOMFLY polynomial of the appropriate link. Extending these results to ribbon graphs is an important open problem.

Finally, recent results combinatorial evaluations of the Tutte and Bollobás-Riordan polynomials [KP] leave an open problem of finding such evaluations for general values of polynomials R_G. It would be interesting to use the Main Theorem to extend the results of [KP]. We leave this direction to the reader.

References

[AFLT] C. Adams, T. Fleming, M. Levin, A. Turner, *Crossing number of alternating knots in $S \times I$*, Pacific J. Math. **203** (2002), 1–22.

[CR]   P. Cotta-Ramusino, M. Rinaldi, *On the algebraic structure of link-diagrams on a 2-dimensional surface*, Comm. Math. Phys. **138** (1991), 137–173.

[B]   B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics **184**, Springer, New York, 1998.

[BR1]   B. Bollobás, O. Riordan, *A Tutte polynomial for colored graphs*, Combinatorics, Probability and Computing **8** (1999), 45–93.

[BR2]   B. Bollobás, O. Riordan, *A polynomial of graphs on orientable surfaces*, Proc. London Math. Soc. **83** (2001), 513–531.

[BR3]   B. Bollobás, O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323**(1) (2002), 81–96.

[DKC]   S. Duzhin, A. Kaishev, S. Chmutov, *The algebra of 3-graphs*, Proc. Steklov Inst. Math. **221** (1998), 157–186.

[DK]   H. A. Dye, L. H. Kauffman, *Minimal surface representations of virtual knots and links*, Preprint \protect\vrule width0pt\protect\href{http://arXiv.org/abs/math/0401035}{arXiv:math.GT/0401035}.

[GT]   J. L. Gross, T. W. Tucker, *Topological graph theory*, Wiley, NY, 1987.

[HP]   J. Hoste, J. H. Przytycki, *An invariant of dichromatic links*, Proc. AMS **105** (1989), 1003–1007.

[IK]   K. Inoue, T. Kaneto, *A Jones type invariant of links in the product space of a surface and the real line*, Journal of Knot Theory and its Ramifications **3** (1994), 153–161.

[J]   F. Jaeger, *Tutte polynomials and link polynomials*, Proc. AMS **103** (1988), 647–654.

[Ka]   N. Kamada, *The crossing number of alternating link diagrams on a surface*, in *Proc. KNOTS'96 (Tokyo)*, 377–382, World Sci. Publ., River Edge, NJ, 1997.

[K1]   L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988), 195–242.

[K2]   L. H. Kauffman, *A Tutte polynomial for signed graphs*, Discrete Appl. Math. **25** (1989), 105–127.

[KP]   M. Korn, I. Pak, *Combinatorial evaluations of the Tutte polynomial*, preprint (2003), available at http://www-math.mit.edu/~pak/research.html.

[M]   V. O. Manturov, *Kauffman-like polynomials and curves in 2-surfaces*, Journal of Knot Theory and its Ramifications **12** (2003), 1145–1153.

[Pr]   J. H. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. **16** (1999) 45–66; Preprint \protect\vrule width0pt\protect\href{http://arXiv.org/abs/math/9809113}{arXiv:math.GT/9809113}

[RT]   N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.

[Tr]   L. Traldi, *A dichromatic polynomials for weighted graphs and link polynomials*, Proc. AMS **106** (1989), 279–286.

[Tu]   V. G. Turaev, *The Conway and Kauffman modules of the solid torus*, J. Soviet Math. **52** (1990), no. 1, 2799–2805.

[W]   D. J. A. Welsh, *Complexity: knots, colourings and counting*, LMS Lecture Note Series **186**, Cambridge U. Press, Cambridge, UK, 1993.

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